

# The $U(N)$ chiral model and exact multi-solitons

Bushra Haider <sup>a, 1</sup> and M. Hassan <sup>a,b, 2</sup>

<sup>a</sup>*Department of Physics,  
University of the Punjab,  
Quaid-e-Azam Campus,  
Lahore-54590, Pakistan.*

<sup>b</sup>*Department of Mathematics,  
University of Glasgow,  
Glasgow G12 8QW, UK.*

## Abstract

We use binary Darboux transformation to obtain exact multi-soliton solutions of principal chiral model and its noncommutative generalization. We also show that the exact multi-solitons of noncommutative principal chiral model in two dimensions and noncommutative (anti-) self dual Yang-Mills equations in four dimensions can be expressed explicitly in terms of quasi-determinants.

PACS: 11.10.Nx, 02.30.Ik

Keywords: Noncommutative geometry, Integrable systems, Principal chiral model

---

<sup>1</sup>bushrahaider@hotmail.com

<sup>2</sup>mhassan@physics.pu.edu.pk, mhassan@maths.gla.ac.uk

## 1 Introduction

It is well known that the classical principal chiral model, a nonlinear sigma model with target manifold being a Lie group, is an integrable field theoretic model in the sense that it contains an infinite sequence of local and non-local conserved quantities and can be embedded into the general scheme of the inverse scattering method [1]-[8]. Recently some investigations have been made regarding the classical integrability of the principal chiral model with and without a Wess-Zumino term and its supersymmetric and noncommutative generalizations [8]-[11]. In these studies [8],[9] the involution of the local conserved quantities amongst themselves and with the non-local conserved quantities has been investigated for the bosonic models (with and without a Wess-Zumino term) and for their supersymmetric generalization. It has been shown in [10] that the supersymmetric generalization of the principal chiral model admits a one-parameter family of superfield flat connections that results in the existence of an infinite number of conservation laws and a superfield Lax formalism.

The study of integrability of principal chiral model is also important from the point of view of integrability and the determination of the exact spectrum of free string theory on  $AdS^5 \times S^5$ . In the last few years a great deal of work has been done in studying the integrability of the classical string sigma model on  $AdS^5 \times S^5$  (see e.g. [42]-[47]). In these studies integrability aspects such as the Lax formalism, the existence of local and nonlocal conserved quantities, the fundamental Poisson bracket algebra and the Yangian symmetry; has been investigated. It has been shown that the infinite number of conserved quantities are in involution for classical string moving on  $\mathbf{R} \times S^3$  submanifold of  $AdS^5 \times S^5$ . In fact, the bosonic strings on  $\mathbf{R} \times S^3$  are described in static gauge by an  $SU(2)$  principal chiral model. From the point of view of string theory, it seems natural to study the construction of multi-soliton solutions for the principal chiral model through a solution generating technique of binary Darboux transformation. The purpose of this work is to use the binary Darboux transformation for obtaining exact multi-soliton solutions of the principal chiral model and the  $N$ -soliton solution has been obtained in terms of ratio of determinants of certain matrices.

The other aspect that we shall be interested in investigating is the generalization of our results to the case of noncommutative principal chiral model. In recent times, there has been an increasing interest in

the study of noncommutative integrable field models (see e.g. [11]-[30]). The interest in these studies is partly due to the fact that the noncommutative field theories play an important role in string theory, D-brane dynamics, quantum Hall effect, etc. (see e.g. [33]-[35]). The noncommutative principal chiral model has been studied recently and it has been shown that the noncommutative generalization of  $U(N)$  principal chiral model is integrable in the sense that it contains an infinite sequence of conserved quantities and it admits a one-parameter family of flat connections leading to a Lax formalism [11]. Moreover, it has been shown that the noncommutative principal chiral model admits a solution generating technique of elementary Darboux transformation which generates noncommutative multi-soliton solutions of the model. In the present work, we shall use the binary Darboux transformation to obtain the exact noncommutative multi-soliton solution of the noncommutative principal chiral model. We show that the exact multi-solitons of noncommutative principal chiral model can be expressed in terms of quasi-determinants introduced by Gelfand and Retakh [36]-[39]. The quasi-determinants also appear in the construction of soliton solutions of some integrable systems (see e.g. [14]-[20]). We also compare our results with those for the noncommutative (anti) self-dual Yang-Mills theory that acts as a master theory in the sense of Ward conjecture [48]-[50] which states that almost all (noncommutative) integrable systems including the principal chiral model can be obtained by (anti) self-dual Yang-Mills equations (or its generalizations) by reduction [31], [32].

## 2 The $U(N)$ principal chiral model

The field variables  $g(x)$  of the  $U(N)$  principal chiral model (PCM) take values in the Lie group  $U(N)$  [1]-[8]. The action for the  $U(N)$  PCM can be written in terms of field  $g(x)$  as <sup>3</sup>

$$S = \frac{1}{2} \int d^2x \text{Tr} (\partial_+ g^{-1} \partial_- g), \quad (2.1)$$

with

$$g^{-1}(x^+, x^-) g(x^+, x^-) = g(x^+, x^-) g^{-1}(x^+, x^-) = 1, \quad (2.2)$$

where  $g(x^+, x^-) \in U(N)$ . The  $U(N)$ -valued field  $g(x^+, x^-)$  can be expressed as

$$g(x^+, x^-) \equiv e^{i\pi_a T^a} = 1 + i\pi_a T^a + \frac{1}{2} (i\pi_a T^a)^2 + \cdots, \quad (2.3)$$

---

<sup>3</sup>Our conventions are such that the two-dimensional coordinates are related as  $x^\pm = \frac{1}{2}(x^0 \pm ix^1)$  and  $\partial^\pm = \frac{1}{2}(\partial_0 \pm i\partial_1)$ .

where  $\pi_a$  is in the Lie algebra  $u(N)$  of the Lie group  $U(N)$  and  $T^a, a = 1, 2, 3, \dots, N^2$ , are Hermitian matrices with the normalization  $\text{Tr}(T^a T^b) = -\delta^{ab}$  and are the generators of  $U(N)$  in the fundamental representation satisfying the algebra

$$[T^a, T^b] = i f^{abc} T^c, \quad (2.4)$$

where  $f^{abc}$  are the structure constants of the Lie algebra  $u(N)$ . For any  $X \in u(N)$ , we write  $X = X^a T^a$  and  $X^a = -\text{Tr}(T^a X)$ . The action (2.1) is invariant under a global continuous symmetry

$$U_L(N) \times U_R(N) : \quad g(x^+, x^-) \mapsto u g v^{-1}, \quad (2.5)$$

where  $u \in U_L(N)$  and  $v \in U_R(N)$ . The Noether conserved currents associated with the global symmetry of the PCM are

$$j_{\pm}^R = -g^{-1}(\partial_{\pm} g), \quad j_{\pm}^L = (\partial_{\pm} g)g^{-1}, \quad (2.6)$$

which take values in the Lie algebra  $u(N)$ , so that one can decompose the currents into components  $j_{\pm}(x^+, x^-) = j_{\pm}^a(x^+, x^-) T^a$ . The equation of motion following from (2.1) corresponds to the conservation of these currents. The left and right currents satisfy the following conservation equation

$$\partial_- j_+ + \partial_+ j_- = 0. \quad (2.7)$$

The currents also obey the zero-curvature condition

$$\partial_- j_+ - \partial_+ j_- + [j_+, j_-] = 0. \quad (2.8)$$

Equations (2.7) and (2.8) can also be expressed as

$$\partial_- j_+ = -\partial_+ j_- = -\frac{1}{2} [j_+, j_-]. \quad (2.9)$$

The equations (2.7)-(2.9) hold for both  $j_{\pm}^L$  and  $j_{\pm}^R$ .

It is well known that the principal chiral model admits a one-parameter family of flat currents [1]. We define a one-parameter family of transformations on the field  $g(x^+, x^-)$  as

$$g \rightarrow g^{(\gamma)} = u^{(\gamma)} g v^{(\gamma)-1}, \quad (2.10)$$

where  $\gamma$  is a parameter and  $u^{(\gamma)}, v^{(\gamma)}$  are the matrices belonging to  $U(N)$ . We choose the boundary values  $u^{(1)} = 1, v^{(1)} = 1$  or  $g^{(1)} = g$ . The matrices  $u^{(\gamma)}$  and  $v^{(\gamma)}$  satisfy the following set of linear equations

$$\partial_{\pm} u^{(\gamma)} = \frac{1}{2} (1 - \gamma^{\mp 1}) j_{\pm}^L u^{(\gamma)}, \quad (2.11)$$

$$\partial_{\pm} v^{(\gamma)} = \frac{1}{2} (1 - \gamma^{\mp 1}) j_{\pm}^R v^{(\gamma)}. \quad (2.12)$$

From now on, we shall consider right-hand currents and drop the superscript  $R$  on the current to simply write  $j_{\pm}^R = j_{\pm}$ . The compatibility condition of the linear system (2.12) is given by

$$\{(1 - \gamma^{-1})\partial_- j_+ - (1 - \gamma)\partial_+ j_- + \left(1 - \frac{1}{2}(\gamma + \gamma^{-1})[j_+, j_-]\right)\}v^{(\gamma)} = 0. \quad (2.13)$$

Under the one-parameter family of transformations, the Noether conserved currents transform as

$$j_{\pm} \mapsto j_{\pm}^{(\gamma)} = \gamma^{\mp 1} v^{(\gamma)-1} j_{\pm} v^{(\gamma)}. \quad (2.14)$$

The linear system (2.12) can also be expressed in the following well known form

$$\partial_{\pm} v(x^+, x^-; \lambda) = A_{\pm}^{(\lambda)} v(x^+, x^-; \lambda), \quad (2.15)$$

where the fields  $A_{\pm}^{(\lambda)}$  are given by

$$A_{\pm}^{(\lambda)} = \mp \frac{\lambda}{1 \mp \lambda} j_{\pm}. \quad (2.16)$$

Here  $\lambda$  is the spectral parameter and is related to  $\gamma$  by  $\lambda = \frac{1-\gamma}{1+\gamma}$ . The compatibility condition of the linear system (2.15) is the zero-curvature condition

$$[\partial_+ - A_+^{(\lambda)}, \partial_- - A_-^{(\lambda)}] \equiv \partial_- A_+^{(\lambda)} - \partial_+ A_-^{(\lambda)} + [A_+^{(\lambda)}, A_-^{(\lambda)}] = 0. \quad (2.17)$$

In other words, we have defined a one-parameter family of connections  $A_{\pm}^{(\lambda)}$  which are flat. The Lax operators

$$L_{\pm}^{(\lambda)} = \partial_{\pm} - A_{\pm}^{(\lambda)}, \quad (2.18)$$

obey the following equations

$$\partial_{\mp} L_{\pm}^{(\lambda)} = [A_{\mp}^{(\lambda)}, L_{\pm}^{(\lambda)}]. \quad (2.19)$$

This is the Lax equation and the set of operators  $(L, A)$  is the given Lax pair of the model [1]-[5]. The Lax formalism detailed above can be used to generate an infinite number of local and non-local conserved quantities and to construct multi-soliton solutions of the model.

### 3 Binary Darboux transformation and exact multi-solitons

The Lax pair of the PCM can be used to construct binary Darboux transformation of the system [40]. We proceed by writing the Lax pair (direct Lax pair) as

$$\begin{aligned}\partial_+ v &= \frac{1}{1-\lambda} j_+ v, \\ \partial_- v &= \frac{1}{1+\lambda} j_- v,\end{aligned}\tag{3.1}$$

where we have used  $\lambda \rightarrow \frac{1}{\lambda}$ . From (3.1), we have

$$\begin{aligned}v^{-1}(\partial_+ v)v^{-1} &= \frac{1}{1-\lambda} v^{-1} j_+ v v^{-1}, \\ \partial_+ v^{-1} &= -\frac{1}{1-\lambda} v^{-1} j_+.\end{aligned}\tag{3.2}$$

Similarly

$$\partial_- v^{-1} = -\frac{1}{1+\lambda} v^{-1} j_-.\tag{3.3}$$

Let us denote

$$v^{-1} \equiv \omega,\tag{3.4}$$

then from equations (3.2), (3.3) and by analogy of direct Lax pair, we define another Lax pair (dual Lax pair) for the matrix field  $\omega$  with spectral parameter  $\lambda'$  as

$$\begin{aligned}\partial_+ \omega &= -\frac{1}{1-\lambda'} \omega j_+, \\ \partial_- \omega &= -\frac{1}{1+\lambda'} \omega j_-.\end{aligned}\tag{3.5}$$

Now consider two solutions  $v_1$  and  $v_2$  of (3.1), then

$$\partial_+ (v_1^{-1} v_2) = (\partial_+ v_1^{-1}) v_2 + v_1^{-1} \partial_+ (v_2),$$

using (3.1) and (3.2) in the above equation we get

$$\partial_+ (v_1^{-1} v_2) = 0.\tag{3.6}$$

Similarly

$$\partial_- (v_1^{-1} v_2) = 0.\tag{3.7}$$

From equations (3.6) and (3.7), we have

$$v_1^{-1} v_2 = C(\lambda),$$

or

$$v_1^{-1} = C(\lambda) v_2^{-1},$$

where  $C(\lambda)$  is some arbitrary matrix function. Now using (3.4), we see that

$$\omega(x, \lambda) = C(\lambda) v^{-1}(x, \lambda), \quad (3.8)$$

where  $\omega(x, \lambda)$  and  $v(x, \lambda)$  are the solutions of the dual and the direct Lax pairs respectively. The matrix field  $g(x)$  can be related to the solution  $\omega$  of the dual Lax pair by

$$g(x) C(0) = \omega(x, \lambda) |_{\lambda=0}. \quad (3.9)$$

Again from (3.1), we see that at  $\lambda = 0$

$$\begin{aligned} \partial_+ v &= j_+ v, \\ (\partial_+ v) v^{-1} &= j_+, \end{aligned}$$

using (2.6) we have

$$(\partial_+ v) v^{-1} = (\partial_+ g) g^{-1}, \quad (3.10)$$

implying

$$g(x) C(0) = v(x, \lambda) |_{\lambda=0}. \quad (3.11)$$

It follows from (3.1) that the matrix function  $v$  may be chosen to satisfy the reality condition

$$v^\dagger(\bar{\lambda}) = v^{-1}(\lambda). \quad (3.12)$$

Similar equation holds for  $\omega$ .

Let  $|m\rangle$  be a column solution and  $\langle n|$  be a row solution of the Lax pairs (3.1), (3.5) with spectral parameters  $\mu$  and  $\nu$  respectively ( $\mu \neq \nu$ ). Through a projection operator  $P$ , the one-fold binary Darboux transformation can be constructed to obtain new matrix solutions  $v[1]$  and  $\omega[1]$  satisfying the direct and

dual Lax pairs (3.1) and (3.5), respectively. The solutions  $v[1]$  and  $\omega[1]$  are related to the old solutions  $v$  and  $\omega$  respectively by the following transformation

$$\begin{aligned} v[1] &= \left( I - \frac{\mu - \nu}{\lambda - \nu} P \right) v, \\ \omega[1] &= \omega \left( I - \frac{\mu - \nu}{\mu - \lambda'} P \right), \end{aligned} \quad (3.13)$$

For the reality condition (3.12) to be satisfied, we have

$$\begin{aligned} \nu &= (\bar{\mu})^{-1}, \\ P^\dagger &= P = P^2, \end{aligned}$$

where the projector  $P$  is defined as

$$P = \frac{|m\rangle \langle n|}{\langle n | m \rangle}, \quad (3.14)$$

with

$$\langle n | m \rangle = \sum_{i=1}^N n_i m_i. \quad (3.15)$$

The projector  $P$  has been expressed in terms of the solutions of Lax pairs (3.1) and (3.5). Let  $g$  be a known solution of the PCM, the binary Darboux transformation gives a new solution  $g[1]$  given by

$$g[1] = \left( I + \frac{\mu - \nu}{\nu} P \right) g, \quad (3.16)$$

where  $v|_{\lambda=0} = g$ . The new solutions  $v[1]$  and  $\omega[1]$  satisfy the direct and dual Lax pairs (3.1) and (3.5) respectively, which shows the covariance of the Lax pair of the PCM under the binary Darboux transformation, implying that the conserved currents  $j_\pm$  transform as

$$\begin{aligned} j_+[1] &= j_+ - (\mu - \nu) \partial_+ P, \\ j_-[1] &= j_- + (\mu - \nu) \partial_- P, \end{aligned} \quad (3.17)$$

Substituting equations (3.13) and (3.17) into the systems we get

$$\begin{aligned} \partial_+ v[1] &= \frac{1}{1 - \lambda} j_+[1] v[1], \\ \partial_- v[1] &= \frac{1}{1 + \lambda} j_-[1] v[1], \end{aligned} \quad (3.18)$$



and

$$\begin{aligned}\partial_+ \omega[1] &= -\frac{1}{1-\lambda'} \omega[1] j_+[1], \\ \partial_- \omega[1] &= -\frac{1}{1+\lambda'} \omega[1] j_-[1].\end{aligned}\tag{3.19}$$

The successive iterations of binary Darboux transformation produces the transformed matrix solutions of direct and dual Lax pairs as

$$\begin{aligned}v[K] &= \left(I - \frac{\mu^{(K)} - \nu^{(K)}}{\lambda - \nu^{(K)}} P[K]\right) \cdots \left(I - \frac{\mu^{(1)} - \nu^{(1)}}{\lambda - \nu^{(1)}} P[1]\right) v, \\ \omega[K] &= \omega \left(I - \frac{\mu^{(1)} - \nu^{(1)}}{\mu^{(1)} - \lambda'} P[1]\right) \cdots \left(I - \frac{\mu^{(K)} - \nu^{(K)}}{\mu^{(K)} - \lambda'} P[K]\right),\end{aligned}\tag{3.20}$$

where

$$P[i] = \frac{|m^{(i)}[i-1]\rangle \langle n^{(i)}[i-1]|}{\langle n^{(i)}[i-1] | m^{(i)}[i-1]\rangle},\tag{3.21}$$

and  $|m^{(i)}[i-1]\rangle$  and  $\langle n^{(i)}[i-1]|$ ,  $(i = 1, 2, 3, \dots, K)$  defined as

$$\begin{aligned}& |m^{(i)}[i-1]\rangle \\ &= \left(I - \frac{\mu^{(i-1)} - \nu^{(i-1)}}{\mu^{(i)} - \nu^{(i-1)}} P[i-1]\right) \cdots \left(I - \frac{\mu^{(i)} - \nu^{(i)}}{\mu^{(i)} - \nu^{(i)}} P[i]\right) |m^{(i)}\rangle, \\ & \langle n^{(i)}[i-1]| \\ &= \langle n^{(i)} | \left(I - \frac{\mu^{(i)} - \nu^{(i)}}{\mu^{(i)} - \nu^{(i)}} P[i]\right) \cdots \left(I - \frac{\mu^{(i-1)} - \nu^{(i-1)}}{\mu^{(i-1)} - \nu^{(i)}} P[i-1]\right),\end{aligned}\tag{3.22}$$

are the matrix-column and matrix-row solutions of direct and dual Lax pairs, with spectral parameters  $\mu^{(i)}$  and  $\nu^{(i)}$  respectively.

If we write the general form of multi-soliton solution of direct Lax pair (3.1) in terms of partial fraction as

$$v[K] = \left(I - \sum_{j=1}^K \frac{R_j}{\lambda - \nu^{(j)}}\right) v,\tag{3.23}$$

and use the fact that  $v[K] = 0$  if  $\lambda = \mu^{(i)}$ ,  $v = |m^{(i)}\rangle$ , we get the  $K$ th iteration formula in the form

$$v[K] = \left(I - \sum_{i,j=1}^K \frac{\mu^{(j)} - \nu^{(i)}}{\lambda - \nu^{(j)}} \frac{|m^{(i)}\rangle \langle n^{(j)}|}{\langle n^{(i)} | m^{(j)}\rangle}\right) v.\tag{3.24}$$

Similarly, by using  $\omega[K] = 0$  for  $\lambda' = \nu^{(i)}$ ,  $\omega = \langle n^{(i)} |$ , we get

$$\omega[K] = \omega \left( I - \sum_{i,j=1}^K \frac{\mu^{(j)} - \nu^{(i)}}{\mu^{(i)} - \lambda'} \frac{|m^{(i)}\rangle \langle n^{(j)}|}{\langle n^{(i)} | m^{(j)}\rangle} \right). \quad (3.25)$$

The relation of  $v[K]$  with the new solution  $g[K]$  of equation (2.7) gives

$$g[K] = \left( I + \sum_{i,j=1}^K \frac{\mu^{(j)} - \nu^{(i)}}{\nu^{(j)}} \frac{|m^{(i)}\rangle \langle n^{(j)}|}{\langle n^{(i)} | m^{(j)}\rangle} \right) g. \quad (3.26)$$

For expression (3.26) to ensure the positive-definite solution of (2.7)

$$\nu^{(i)} = \left( \bar{\mu}^{(i)} \right)^{-1}, \quad (3.27)$$

$$\langle n^{(i)} | = \left( |m^{(i)}\rangle \right)^\dagger g^{-1} = \langle m^{(i)} | g^{-1}. \quad (3.28)$$

The solutions  $v[K]$  and  $\omega[K]$  expressed in additive form (3.24) and (3.25) respectively, are subjected to the reality condition (3.12). The condition (3.12) to be satisfied, one requires that the projectors  $P[i]$  be Hermitian and mutually orthogonal, i.e.

$$\begin{aligned} P^\dagger[i] &= P[i] = P^2[i], \\ P[i] P[j] &= 0, \quad \text{for } i \neq j. \end{aligned} \quad (3.29)$$

The solutions of the  $U(N)$  principal chiral model obtained here are same as the solutions obtained through the well known dressing method of Zakharov and Shabat [2], [5]. In the dressing method, the solution of the system is obtained by reducing the solution of the spectral problem to that of a Riemann-Hilbert problem with zero. By using the technique of complex analysis, the solution of the system is expressed in terms of projectors that relate solutions of Riemann-Hilbert problem in a simple algebraic form. In the binary Darboux transformation method discussed here, we express the solution of the system in terms of projectors that can be obtained in terms of the solutions of the direct and dual Lax pairs of the system. We combine two elementary Darboux transformations to construct binary Darboux transformation that generates solutions of the system in terms of given projectors. Let us now consider the second iteration of binary Darboux transformation. Take  $\mu^{(1)} = \mu$ ,  $\mu^{(2)} = \bar{\mu}^{-1} + \varepsilon$ , and  $\langle m^{(1)} g^{-1} | m^{(2)} \rangle = O(\varepsilon)$ . Taking the

limit  $\varepsilon \rightarrow 0$ , the coefficients of matrix  $g[2]$  become

$$g[2]_{ik} = \frac{\begin{vmatrix} g_{ik} & -\langle m^{(1)}g^{-1} | g^{(k)} \rangle & -\langle m^{(2)}g^{-1} | g^{(k)} \rangle \\ m_i^{(1)} & \frac{\langle m^{(1)}g^{-1} | m^{(1)} \rangle}{|\mu|^2-1} & \mu^{-1} \langle A^\dagger g^{-1} | m^{(1)} \rangle \\ m_i^{(2)} & \bar{\mu}^{-1} \langle m^{(1)}g^{-1} | A \rangle & \frac{|\mu|^2}{1-|\mu|^2} \langle m^{(2)}g^{-1} | m^{(2)} \rangle \end{vmatrix}}{\begin{vmatrix} \frac{\langle m^{(1)}g^{-1} | m^{(1)} \rangle}{|\mu|^2-1} & \mu^{-1} \langle A^\dagger g^{-1} | m^{(1)} \rangle \\ \bar{\mu}^{-1} \langle m^{(1)}g^{-1} | A \rangle & \frac{|\mu|^2}{1-|\mu|^2} \langle m^{(2)}g^{-1} | m^{(2)} \rangle \end{vmatrix}}, \quad (3.30)$$

where

$$\begin{aligned} A &= \frac{\partial |m^{(2)}\rangle}{\partial \mu^{(2)}} \Big|_{\mu^{(2)}=\bar{\mu}^{-1}}, \\ g &= \left( g^{(1)}, \dots, g^{(N)} \right), \\ |m^{(i)}\rangle &= \left( m_i^{(i)}, \dots, m_N^{(i)} \right)^T, \quad i = 1, 2, \dots, K. \end{aligned}$$

Similarly applying binary Darboux transformation  $K$  times, we see that the coefficients of matrix  $g[K]$  are

$$\begin{aligned} &g[K]_{ik} \\ &= \frac{\begin{vmatrix} g_{ik} & -a_{1k} & -a_{2k} & -a_{3k} & -a_{4k} & \cdots & -a_{Kk} \\ m_i^{(1)} & \frac{1}{|\mu|^2-1} b_{11} & \mu^{-1} A_{11}^\dagger & \frac{1}{|\mu|^2-1} A_{21}^\dagger & \mu^{-1} A_{31}^\dagger & \cdots & c_{1K} A_{(K-1)1}^\dagger \\ m_i^{(2)} & \bar{\mu}^{-1} A_{11} & \frac{|\mu|^2}{1-|\mu|^2} b_{22} & \bar{\mu}^{-1} A_{31} & \frac{|\mu|^2}{1-|\mu|^2} A_{41} & \cdots & c_{2K} A_{K1} \\ m_i^{(3)} & \frac{1}{|\mu|^2-1} A_{12} & \mu^{-1} A_{22} & \frac{1}{|\mu|^2-1} b_{33} & \mu^{-1} A_{42} & \cdots & c_{3K} A_{K2} \\ m_i^{(4)} & \bar{\mu}^{-1} A_{13} & \frac{|\mu|^2}{1-|\mu|^2} A_{23} & \bar{\mu}^{-1} A_{33} & \frac{|\mu|^2}{1-|\mu|^2} b_{44} & \cdots & c_{4K} A_{K3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ m_i^{(K)} & c_{K1} A_{1(K-1)} & c_{K2} A_{2(K-1)} & c_{K3} A_{3(K-1)} & c_{K4} A_{4(K-1)} & \cdots & c_{kK} b_{KK} \end{vmatrix}}{\begin{vmatrix} \frac{1}{|\mu|^2-1} b_{11} & \mu^{-1} A_{11}^\dagger & \frac{1}{|\mu|^2-1} A_{21}^\dagger & \mu^{-1} A_{31}^\dagger & \cdots & c_{1K} A_{(K-1)1}^\dagger \\ \bar{\mu}^{-1} A_{11} & \frac{|\mu|^2}{1-|\mu|^2} b_{22} & \bar{\mu}^{-1} A_{31} & \frac{|\mu|^2}{1-|\mu|^2} A_{41} & \cdots & c_{2K} A_{K1} \\ \frac{1}{|\mu|^2-1} A_{12} & \mu^{-1} A_{22} & \frac{1}{|\mu|^2-1} b_{33} & \mu^{-1} A_{42} & \cdots & c_{3K} A_{K2} \\ \bar{\mu}^{-1} A_{13} & \frac{|\mu|^2}{1-|\mu|^2} A_{23} & \bar{\mu}^{-1} A_{33} & \frac{|\mu|^2}{1-|\mu|^2} b_{44} & \cdots & c_{4K} A_{K3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{K1} A_{1(K-1)} & c_{K2} A_{2(K-1)} & c_{K3} A_{3(K-1)} & c_{K4} A_{4(K-1)} & \cdots & c_{kK} b_{KK} \end{vmatrix}}, \quad (3.31) \end{aligned}$$

where

$$a_{ik} = \langle m^{(i)}g^{-1} | g^{(k)} \rangle,$$

$$\begin{aligned}
b_{ik} &= \left\langle m^{(i)} g^{-1} \mid m^{(k)} \right\rangle, \\
A_{ik} &= \left\langle m^{(i)} g^{-1} \mid A_k \right\rangle, \\
A_{ik}^\dagger &= \left\langle A_i^\dagger g^{-1} \mid m^{(k)} \right\rangle.
\end{aligned}$$

Please note that we have used the following notation for the coefficients of the entries in  $K$ -th column of  $g[K]_{ik}$

$$c_{ij} = \begin{cases} \mu^{-1} & (\text{for odd values of } i, j), \\ \bar{\mu}^{-1} & (\text{for even values of } i, j), \\ \frac{1}{|\mu|^2 - 1} & (\text{for } i \text{ odd, } j \text{ even}), \\ \frac{|\mu|^2}{1 - |\mu|^2} & (\text{for } i \text{ even, } j \text{ odd}), \end{cases} \quad (3.32)$$

while the entries in  $K$ -th row of  $g[K]_{ik}$  are obtained as

$$c_{ji} = \bar{c}_{ij}. \quad (3.33)$$

The solution  $g[K]_{ik}$  is the  $K$ -soliton solution of the  $U(N)$  principal chiral model. It has been mentioned earlier that such solutions can be constructed using the dressing method.

#### 4 Binary Darboux transformation for a noncommutative $U(N)$ principal chiral model

In this section, we study the binary Darboux transformation of a noncommutative  $U(N)$  principal chiral model and obtain the multi-soliton solutions of the model in terms of quasi-determinants. In recent years, a lot of investigations have been made regarding the noncommutative generalization of integrable models (see e.g. [11]-[30]). In reference [24], a noncommutative extension of  $U(N)$ -principal chiral model (nc-PCM) has been presented and it is concluded that this noncommutative extension gives no extra constraints for the theory to be integrable. The non-local conserved quantities of nc-PCM have also been derived using the iterative method of Brézin-Itzykson-Zinn-Justin-Zuber (BIZZ) [3] and the Lax formalism of the nc-PCM to derive conserved quantities has been developed in [11].

One way of obtaining noncommutative field theories is by the replacement of ordinary products of field functions in commutative field theories with their star products ( $\star$ -products) and the resulting theories are realized as deformed theories from the commutative ones. The  $\star$ -product is defined for ordinary fields on

flat spaces, explicitly by [41]

$$\begin{aligned} f(x)g(x) &\rightarrow (f \star g)(x) = \exp\left[\left(\frac{i}{2}\theta^{\mu\nu}\partial_\mu^{x_1}\partial_\nu^{x_2}\right)f(x_1)g(x_2)\big|_{x_1=x_2=x}\right] \\ &= f(x)g(x) + \frac{i\theta^{\mu\nu}}{2}\partial_\mu f(x)\partial_\nu g(x) + O(\theta^2), \end{aligned} \quad (4.1)$$

where  $\partial_\mu^{x_i} = \frac{\partial}{\partial x_i^\mu}$ . The noncommutativity of coordinates of the Euclidean space  $R^D$  is defined as

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \quad (4.2)$$

where  $\theta^{\mu\nu}$  is the second rank antisymmetric real constant tensor known as the deformation parameter. The  $\star$ -product of functions carries intrinsically the noncommutativity of the coordinates and is associative i.e.

$$(f \star g) \star h = f \star (g \star h)$$

As can be seen from (4.1), the noncommutative field theories reduce to the ordinary (commutative) field theories as the deformation parameter  $\theta$  goes to zero (for more details see e.g. [33]-[35]). Following section 2, we define the action for the two-dimensional  $U(N)$  noncommutative principal chiral model (nc-PCM) as [11]

$$S^\star = \frac{1}{2} \int d^2x \text{Tr} (\partial_+ g^{-1} \star \partial_- g), \quad (4.3)$$

with

$$g^{-1}(x^+, x^-) \star g(x^+, x^-) = g(x^+, x^-) \star g^{-1}(x^+, x^-) = 1, \quad (4.4)$$

where  $g(x^+, x^-) \in U(N)$ . In this case, the  $U(N)$ -valued field  $g(x^+, x^-)$  is defined as

$$g(x^+, x^-) \equiv e_\star^{i\pi_a T^a} = 1 + i\pi_a T^a + \frac{1}{2}(i\pi_a T^a)_\star^2 + \dots, \quad (4.5)$$

The action (4.3) is invariant under a global continuous symmetry

$$U_L(N) \times U_R(N) : \quad g(x^+, x^-) \mapsto u \star g \star v^{-1}, \quad (4.6)$$

and the corresponding Noether conserved currents of the nc-PCM are

$$j_\pm^{\star R} = -g^{-1} \star (\partial_\pm g), \quad j_\pm^{\star L} = (\partial_\pm g) \star g^{-1}, \quad (4.7)$$

which take values in the Lie algebra  $u(N)$ . The left and right currents satisfy the following conservation equation

$$\partial_- j_+^* + \partial_+ j_-^* = 0, \quad (4.8)$$

and the zero-curvature condition

$$\partial_- j_+^* - \partial_+ j_-^* + [j_+^*, j_-^*]_* = 0, \quad (4.9)$$

where  $[\cdot, \cdot]_*$  is the commutator with respect to  $\star$ -product i.e. for any functions  $f$  and  $g$ ,  $[f, g]_* = f \star g - g \star f$ . It can be easily seen that the equations (4.8) and (4.9) appear as the compatibility condition of the following set of linear equations (Lax pair)

$$\partial_{\pm} v(x^+, x^-; \lambda) = A_{\pm}^{*(\lambda)} \star v(x^+, x^-; \lambda), \quad (4.10)$$

where the fields  $A_{\pm}^{*(\lambda)}$  are given by

$$A_{\pm}^{*(\lambda)} = \mp \frac{\lambda}{1 \mp \lambda} j_{\pm}^*, \quad (4.11)$$

and  $\lambda$  is the spectral parameter.

By analogy of section 3, the Lax pair of the nc-PCM can be used to construct binary Darboux transformation of the system. We proceed by rewriting the Lax pair (direct Lax pair) as

$$\partial_+ v = \frac{1}{1 - \lambda} j_+^* \star v, \quad (4.12)$$

$$\partial_- v = \frac{1}{1 + \lambda} j_-^* \star v. \quad (4.13)$$

The dual Lax pair for the matrix field  $\omega$  with spectral parameter  $\lambda'$  is given as

$$\partial_+ \omega = -\frac{1}{1 - \lambda'} \omega \star j_+^*, \quad (4.14)$$

$$\partial_- \omega = -\frac{1}{1 + \lambda'} \omega \star j_-^*. \quad (4.15)$$

Since all the objects involved are of matrix nature, therefore, the binary Darboux transformation can be constructed for the nc-PCM in the same way as for the usual (commutative) PCM. Following the previous section, one arrives at the transformation

$$v[1] = \left( I - \frac{\mu - \nu}{\lambda - \nu} P \right) \star v,$$

$$\begin{aligned}
\omega[1] &= \omega \star \left( I - \frac{\mu - \nu}{\mu - \lambda'} P \right), \\
g^*[1] &= \left( I + \frac{\mu - \nu}{\nu} P \right) \star g^*,
\end{aligned} \tag{4.16}$$

where the  $P$  is the projector, defined as

$$P = |m\rangle \star (\langle m | n \rangle)^{-1} \star \langle n|. \tag{4.17}$$

By applying the successive Darboux transformation on  $v, \omega$  and  $g$ , we arrive at the following solution

$$\begin{aligned}
g^*[K] &= [I + (\sum_{i,j=1}^K \frac{\mu^{(j)} - \nu^{(i)}}{\nu^{(j)}} \\
&\times |m^{(i)}\rangle \star (\langle n^{(i)} | m^{(j)} \rangle)^{-1} \star \langle n^{(j)}|)] \star g^*,
\end{aligned} \tag{4.18}$$

where

$$\nu^{(i)} = (\bar{\mu}^{(i)})^{-1}, \tag{4.19}$$

$$\langle n^{(i)}| = (|m^{(i)}\rangle)^{\dagger} \star (g^*)^{-1} = \langle m^{(i)}| \star (g^*)^{-1}, \tag{4.20}$$

The solution (4.18) of nc-PCM is different from (3.26) of the usual PCM in the sense that the product of functions has been replaced with the corresponding  $\star$ -product. The difference will clearly show up when we take explicit expressions of solution (4.18) of nc-PCM. In the usual (commutative) case the solution appears as a ratio of determinants of certain functions but in the noncommutative case, the solution appears as quasi-determinant. To show this we consider the second iteration of the binary Darboux transformation. Take  $\mu^{(1)} = \mu, \mu^{(2)} = \bar{\mu}^{-1} + \varepsilon$ , and  $\langle m^{(i)}(g^*)^{-1} | m^{(j)} \rangle = O(\varepsilon)$ . Taking the limit  $\varepsilon \rightarrow 0$ , the coefficients of matrix  $g^*[2]$  may then be written in terms of quasi-determinant<sup>4</sup> as

$$g^*[2]_{ik} = \begin{vmatrix} \boxed{g_{ik}^*} & -\langle m^{(1)}(g^*)^{-1} | g^{*(k)} \rangle & -\langle m^{(2)}(g^*)^{-1} | g^{*(k)} \rangle \\ m_i^{(1)} & \frac{\langle m^{(1)}(g^*)^{-1} | m^{(1)} \rangle}{|\mu|^2 - 1} & \mu^{-1} \langle A^{\dagger}(g^*)^{-1} | m^{(1)} \rangle \\ m_i^{(2)} & \bar{\mu}^{-1} \langle m^{(1)}(g^*)^{-1} | A \rangle & \frac{|\mu|^2}{1 - |\mu|^2} \langle m^{(2)}(g^*)^{-1} | m^{(2)} \rangle \end{vmatrix}, \tag{4.21}$$

<sup>4</sup>Let  $A = (a_{ij})$  be a  $N \times N$  matrix and  $B = (b_{ij})$  be the inverse matrix of  $A$ , that is,  $A \star B = B \star A = 1$ . Quasi-determinants of  $A$  are defined formally as the inverse of the elements of  $B = A^{-1} : |A_{ij}| = b^{-1}$ . Quasi-determinants can be also given iteratively by:

$$|A|_{ij} = a_{ij} - \sum a_{ip} \star |A_{ij}|_{pq}^{-1} \star a_{qj}.$$

The quasi-determinant for a  $1 \times 1$  matrix  $A = (a_{11})$  is

$$|A|_{11} = a_{11}$$

where

$$A = \frac{\partial |m^{(2)}\rangle}{\partial \mu^{(2)}} \big|_{\mu^{(2)}=\bar{\mu}^{-1}},$$

$$g^* = \left( g^{*(1)}, \dots, g^{*(N)} \right).$$

The  $K$ -th iteration of BDT leads to the coefficients of matrix  $g[K]$  as

$$g^*[K]_{ik} = \begin{vmatrix} \boxed{g_{ik}^*} & a_{1k}^* & a_{2k}^* & a_{3k}^* & a_{4k}^* & \cdots & a_{Kk}^* \\ m_i^{(1)} & \frac{1}{|\mu|^2-1} b_{11}^* & \mu^{-1} A_{11}^{\star\dagger} & \frac{1}{|\mu|^2-1} A_{21}^{\star\dagger} & \mu^{-1} A_{31}^{\star\dagger} & \cdots & c_{1K} A_{(K-1)1}^{\star\dagger} \\ m_i^{(2)} & \bar{\mu}^{-1} A_{11}^* & \frac{|\mu|^2}{1-|\mu|^2} b_{22}^* & \bar{\mu}^{-1} A_{31}^* & \frac{|\mu|^2}{1-|\mu|^2} A_{41}^* & \cdots & c_{2K} A_{K1}^* \\ m_i^{(3)} & \frac{1}{|\mu|^2-1} A_{12}^* & \mu^{-1} A_{22}^* & \frac{1}{|\mu|^2-1} b_{33}^* & \mu^{-1} A_{42}^* & \cdots & c_{3K} A_{K2}^* \\ m_i^{(4)} & \bar{\mu}^{-1} A_{13}^* & \frac{|\mu|^2}{1-|\mu|^2} A_{23}^* & \bar{\mu}^{-1} A_{33}^* & \frac{|\mu|^2}{1-|\mu|^2} b_{44}^* & \cdots & c_{4K} A_{K3}^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ m_i^{(K)} & c_{K1} A_{1(K-1)}^* & c_{K2} A_{2(K-1)}^* & c_{K3} A_{3(K-1)}^* & c_{K4} A_{4(K-1)}^* & \cdots & c_{KK} b_{KK}^* \end{vmatrix}, \quad (4.22)$$

where

$$a_{ik}^* = - \left\langle m^{(i)}(g^*)^{-1} \mid g^{*(k)} \right\rangle,$$

$$b_{ik}^* = \left\langle m^{(i)}(g^*)^{-1} \mid m^{(k)} \right\rangle,$$

$$A_{ik}^* = \left\langle m^{(i)}(g^*)^{-1} \mid A_k \right\rangle,$$

$$A_{ik}^{\star\dagger} = \left\langle A_i^\dagger(g^*)^{-1} \mid m^{(k)} \right\rangle,$$

$$\mu^{(1)} = \mu,$$

---

For a  $2 \times 2$  matrix  $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ , there exist four quasi-determinants given as

$$|A|_{11} = \begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12} \star a_{22}^{-1} \star a_{21},$$

$$|A|_{21} = \begin{vmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & a_{22} \end{vmatrix} = a_{21} - a_{22} \star a_{12}^{-1} \star a_{11},$$

$$|A|_{12} = \begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{vmatrix} = a_{12} - a_{11} \star a_{21}^{-1} \star a_{22},$$

$$|A|_{22} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix} = a_{22} - a_{21} \star a_{11}^{-1} \star a_{12}.$$

For more examples and properties of quasi determinants see[36]-[39].



$$\begin{aligned}
\mu^{(i)} &= \frac{1}{\bar{\mu}^{(i-1)}} + \varepsilon, \quad i = 2, 3, \dots, (N-1), \\
\langle m^{(i)}(g^\star)^{-1} | m^{(j)} \rangle &= O(\varepsilon), \quad i \neq j, \\
A_{i-1} &= \frac{\partial |m^{(i)}\rangle}{\partial \mu^{(i)}} \big|_{\mu^{(i)}=\bar{\mu}^{-1}}.
\end{aligned}$$

The coefficients of the entries in  $K$ -th row and column of  $g^\star [K]_{ik}$  are same as given in equations (3.32) and (3.33). Note that in the commutative limit i.e.  $\theta \rightarrow 0$ , the quasi-determinants (4.21) and (4.22) reduce to the ratio of determinants (3.30) and (3.31) respectively.

## 5 Relation to binary Darboux transformation for noncommutative (anti) self-dual Yang-Mills equations

(Anti) self-dual Yang-Mills ((A)SDYM) theory is a well known example of multi-dimensional integrable systems [48]-[50]. The (A)SDYM equations also act as master equations of many integrable equations in the sense of Ward conjecture [48]-[50]. The noncommutative generalization of (A)SDYM equations and its integrability aspects have been investigated recently (e.g. [21]-[23]). Following [23], we write the (A)SDYM equations on a four-dimensional noncommutative space<sup>5</sup>

$$\begin{aligned}
\partial_{\bar{y}} \mathcal{J}_y^\star + \partial_{\bar{z}} \mathcal{J}_z^\star &= 0, \\
\partial_y \mathcal{J}_{\bar{y}}^\star + \partial_z \mathcal{J}_{\bar{z}}^\star &= 0,
\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}
\mathcal{J}_{\bar{y}}^\star &= \partial_{\bar{y}} J^\star \star (J^\star)^{-1}, \\
\mathcal{J}_{\bar{z}}^\star &= \partial_{\bar{z}} J^\star \star (J^\star)^{-1}
\end{aligned}$$

---

<sup>5</sup>The coordinates  $x_\mu, \mu = 0, 1, 2, 3$  on 4-dimensional noncommutative Euclidean space  $E^4$  are related to the coordinates on noncommutative complex Euclidean space as

$$\begin{aligned}
y &= x_0 + ix_3, & \bar{y} &= x_0 - ix_3, \\
z &= x_1 + ix_2, & \bar{z} &= x_1 - ix_2.
\end{aligned}$$

The Yang-Mills fields are  $N \times N$  matrix-valued 1-forms representing  $U(N)$  connections with components

$$\begin{aligned}
A_y^\star &= g^{-1} \star \partial_y g, & A_{\bar{y}}^\star &= \bar{g}^{-1} \star \partial_{\bar{y}} \bar{g}, \\
A_z^\star &= g^{-1} \star \partial_z g, & A_{\bar{z}}^\star &= \bar{g}^{-1} \star \partial_{\bar{z}} \bar{g},
\end{aligned}$$

where  $g, \bar{g}$  and their inverses with respect to  $\star$ -product  $g^{-1}, \bar{g}^{-1}$  are functions of  $y, \bar{y}, z, \bar{z}$  and are matrices belonging to  $U(N)$ .

and  $(J^\star)^{-1} = \bar{g}^{-1} \star g$  is the inverse of  $J^\star$  with respect to the  $\star$ -product. The noncommutative (A)SDYM equations (nc-(A)SDYM equations) can also be expressed as the compatibility condition of the following linear system (Lax pair)

$$\begin{aligned} (\partial_y + \lambda \partial_{\bar{z}}) \Psi(y, \bar{y}, z, \bar{z}; \lambda) &= \mathcal{J}_y^\star \star \Psi(y, \bar{y}, z, \bar{z}; \lambda), \\ (\partial_z - \lambda \partial_{\bar{y}}) \Psi(y, \bar{y}, z, \bar{z}; \lambda) &= \mathcal{J}_z^\star \star \Psi(y, \bar{y}, z, \bar{z}; \lambda), \end{aligned} \quad (5.2)$$

where  $\Psi(y, \bar{y}, z, \bar{z}; \lambda)$  is some  $N \times N$  matrix-valued field and  $\lambda$  is the spectral parameter. The compatibility of the linear system (5.2) is

$$(\partial_z \mathcal{J}_y^\star - \partial_y \mathcal{J}_z^\star + [\mathcal{J}_y^\star, \mathcal{J}_z^\star]) - \lambda (\partial_{\bar{y}} \mathcal{J}_y^\star + \partial_{\bar{z}} \mathcal{J}_z^\star) = 0.$$

Following [23], it is easy to see that nc-(A)SDYM equations (5.1) reduce to a noncommutative two-dimensional principal chiral field equation, if we take  $y = \bar{y} = x_0$  and  $z = \bar{z} = x_1$ ,

$$\partial_0 \mathcal{J}_0^\star + \partial_1 \mathcal{J}_1^\star = 0,$$

where  $\mathcal{J}_0^\star$  and  $\mathcal{J}_1^\star$  are the components of conserved currents associated with the global transformation  $U(N)_L \times U(N)_R$ .

The binary Darboux transformation of nc-(A)SDYM equation has been studied in [23] where the  $K$ -soliton solution  $J[K]$  has been expressed in terms of the projector  $P[i]$  as

$$J^\star[K] = \left( I + \frac{\mu^{(K)} - \nu^{(K)}}{\nu^{(K)}} P[N] \right) \star \cdots \star \left( I + \frac{\mu^{(1)} - \nu^{(1)}}{\nu^{(1)}} P[1] \right) \star J^\star,$$

where

$$P[i] = \left\langle \psi^{(i)}[i-1] \middle| \star \left\langle \phi^{(i)}[i-1] \middle| \psi^{(i)}[i-1] \right\rangle^{-1} \star \left| \phi^{(i)}[i-1] \right\rangle \right\rangle,$$

and  $\langle \psi^{(i)}[i-1] |$  and  $|\phi^{(i)}[i-1]\rangle$  are row and column solutions of the direct and dual Lax pairs of nc-(A)SDYM equations with spectral parameters  $\mu^{(i)}$  and  $\nu^{(i)}$  respectively. The  $K$ -soliton solution of nc-(A)SDYM equations can also be expressed in terms of quasi-determinants as in the case with nc-PCM.

For this, we write (see [40])

$$J^\star [2]_{ik} = \begin{vmatrix} \boxed{J_{ik}^\star} & -\langle \psi^{(1)}(J^\star)^{-1} \mid J^{\star(k)} \rangle & -\langle \psi^{(2)}(J^\star)^{-1} \mid J^{\star(k)} \rangle \\ \psi_i^{(1)} & \frac{\langle \psi^{(1)}(J^\star)^{-1} \mid \psi^{(1)} \rangle}{|\mu|^2 - 1} & \mu^{-1} \langle A'^\dagger (J^\star)^{-1} \mid \psi^{(1)} \rangle \\ \psi_i^{(2)} & \bar{\mu}^{-1} \langle \psi^{(1)}(J^\star)^{-1} \mid A' \rangle & \frac{|\mu|^2}{1 - |\mu|^2} \langle \psi^{(2)}(J^\star)^{-1} \mid \psi^{(2)} \rangle \end{vmatrix},$$

where

$$\begin{aligned} \nu^{(i)} &= \left( \bar{\mu}^{(i)} \right)^{-1}, \\ \langle \phi^{(i)} \mid &= \left( \left| \psi^{(i)} \right\rangle \right)^\dagger \star (J^\star)^{-1} = \langle \psi^{(i)} \mid \star (J^\star)^{-1}, \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} A' &= \frac{\partial \left| \psi^{(2)} \right\rangle}{\partial \mu^{(2)}} \Big|_{\mu^{(2)} = \bar{\mu}^{-1}}, \\ J^\star &= \left( J^{\star(1)}, \dots, J^{\star(N)} \right), \\ \left| \psi^{(i)} \right\rangle &= \left( \psi_1^{(i)}, \dots, \psi_N^{(i)} \right)^T, \quad i = 1, 2, \dots, K. \end{aligned}$$

The  $J^\star [K]$  is now given as

$$\begin{aligned}
& J^\star [K]_{ik} \\
& \left| \begin{array}{cccccc}
\boxed{J_{ik}^\star} & a_{1k}'^\star & a_{2k}'^\star & a_{3k}'^\star & a_{4k}'^\star & \cdots & a_{Kk}'^\star \\
\psi_i^{(1)} & \frac{1}{|\mu|^2-1} b_{11}'^\star & \mu^{-1} A_{11}'^{\star\dagger} & \frac{1}{|\mu|^2-1} A_{21}'^{\star\dagger} & \mu^{-1} A_{31}'^{\star\dagger} & \cdots & c_{1K} A_{(K-1)1}'^{\star\dagger} \\
\psi_i^{(2)} & \bar{\mu}^{-1} A_{11}'^\star & \frac{|\mu|^2}{1-|\mu|^2} b_{22}'^\star & \bar{\mu}^{-1} A_{31}'^\star & \frac{|\mu|^2}{1-|\mu|^2} A_{41}'^\star & \cdots & c_{2K} A_{K1}'^\star \\
\psi_i^{(3)} & \frac{1}{|\mu|^2-1} A_{12}'^\star & \mu^{-1} A_{22}'^\star & \frac{1}{|\mu|^2-1} b_{33}'^\star & \mu^{-1} A_{42}'^\star & \cdots & c_{3K} A_{K2}'^\star \\
\psi_i^{(4)} & \bar{\mu}^{-1} A_{13}'^\star & \frac{|\mu|^2}{1-|\mu|^2} A_{23}'^\star & \bar{\mu}^{-1} A_{33}'^\star & \frac{|\mu|^2}{1-|\mu|^2} b_{44}'^\star & \cdots & c_{4K} A_{K3}'^\star \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\psi_i^{(K)} & c_{K1} A_{1(K-1)}'^\star & c_{K2} A_{2(K-1)}'^\star & c_{K3} A_{3(K-1)}'^\star & c_{K4} A_{4(K-1)}'^\star & \cdots & c_{KK} b_{KK}'^\star
\end{array} \right|,
\end{aligned}$$

where

$$\begin{aligned}
a_{ik}'^\star &= -\left\langle \psi^{(i)}(J^\star)^{-1} \mid J^{\star(k)} \right\rangle, \\
b_{ik}'^\star &= \left\langle \psi^{(i)}(J^\star)^{-1} \mid \psi^{(k)} \right\rangle, \\
A_{ik}'^\star &= \left\langle \psi^{(i)}(J^\star)^{-1} \mid A_k' \right\rangle, \\
A_{ik}'^{\star\dagger} &= \left\langle A_i'^{\dagger}(J^\star)^{-1} \mid \psi^{(k)} \right\rangle, \\
\mu^{(1)} &= \mu, \\
\mu^{(i)} &= \frac{1}{\bar{\mu}^{(i-1)}} + \varepsilon, \quad i = 2, 3, \dots, (K-1), \\
\left\langle \psi^{(i)}(J^\star)^{-1} \mid \psi^{(j)} \right\rangle &= O(\varepsilon), \quad i \neq j, \\
A_{i-1}' &= \frac{\partial |\psi^{(i)}\rangle}{\partial \mu^{(i)}} \Big|_{\mu^{(i)} = \bar{\mu}^{-1}}.
\end{aligned}$$

The coefficients of the entries in  $K$ -th row and column of  $J^\star [K]_{ik}$  are same as given in equations (3.32) and (3.33). Here again, one can see that in the commutative limit i.e.  $\theta \rightarrow 0$ , the quasi-determinants of multi-solitons reduce to the ratio of determinants of multi-solitons of usual (commutative) (A)SDYM

equations. So in the limit  $\theta \rightarrow 0$ , we get the following two-soliton solution as

$$J^* [2]_{ik} \rightarrow J [2]_{ik} = \frac{\begin{vmatrix} J_{ik} & -\langle \psi^{(1)} J^{-1} | J^{(k)} \rangle & \langle \psi^{(2)} J^{-1} | J^{(k)} \rangle \\ \psi_i^{(1)} & \frac{\langle \psi^{(1)} J^{-1} | \psi^{(1)} \rangle}{|\mu|^2 - 1} & \mu^{-1} \langle A'^{\dagger} J^{-1} | \psi^{(1)} \rangle \\ \psi_i^{(2)} & \bar{\mu}^{-1} \langle \psi^{(1)} J^{-1} | A' \rangle & \frac{|\mu|^2}{1 - |\mu|^2} \langle \psi^{(2)} J^{-1} | \psi^{(2)} \rangle \end{vmatrix}}{\begin{vmatrix} \frac{\langle \psi^{(1)} J^{-1} | \psi^{(1)} \rangle}{|\mu|^2 - 1} & \mu^{-1} \langle A'^{\dagger} J^{-1} | \psi^{(1)} \rangle \\ \bar{\mu}^{-1} \langle \psi^{(1)} J^{-1} | A' \rangle & \frac{|\mu|^2}{1 - |\mu|^2} \langle \psi^{(2)} J^{-1} | \psi^{(2)} \rangle \end{vmatrix}},$$

and similarly for the multi-soliton solution, we get

$$\begin{aligned}
& J^\star [K]_{ik} \rightarrow J [K]_{ik} \\
& \left( \begin{array}{cccccc}
J_{ik} & -a'_{1k} & -a'_{2k} & -a'_{3k} & -a'_{4k} & \cdots & -a'_{Kk} \\
\psi_i^{(1)} & \frac{1}{|\mu|^2-1} b'_{11} & \mu^{-1} A'_{11}^\dagger & \frac{1}{|\mu|^2-1} A'_{21}^\dagger & \mu^{-1} A'_{31}^\dagger & \cdots & c_{1K} A'_{(K-1)1}^\dagger \\
\psi_i^{(2)} & \bar{\mu}^{-1} A'_{11} & \frac{|\mu|^2}{1-|\mu|^2} b'_{22} & \bar{\mu}^{-1} A'_{31} & \frac{|\mu|^2}{1-|\mu|^2} A'_{41} & \cdots & c_{2K} A'_{K1} \\
\psi_i^{(3)} & \frac{1}{|\mu|^2-1} A'_{12} & \mu^{-1} A'_{22} & \frac{1}{|\mu|^2-1} b'_{33} & \mu^{-1} A'_{42} & \cdots & c_{3K} A'_{K2} \\
\psi_i^{(4)} & \bar{\mu}^{-1} A'_{13} & \frac{|\mu|^2}{1-|\mu|^2} A'_{23} & \bar{\mu}^{-1} A'_{33} & \frac{|\mu|^2}{1-|\mu|^2} b'_{44} & \cdots & c_{4K} A'_{K3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\psi_i^{(K)} & c_{K1} A'_{1(K-1)} & c_{K2} A'_{2(K-1)} & c_{K3} A'_{3(K-1)} & c_{K4} A'_{4(K-1)} & \cdots & c_{KK} b'_{KK}
\end{array} \right) \\
= & \left( \begin{array}{cccccc}
\frac{1}{|\mu|^2-1} b'_{11} & \mu^{-1} A'_{11}^\dagger & \frac{1}{|\mu|^2-1} A'_{21}^\dagger & \mu^{-1} A'_{31}^\dagger & \cdots & c_{1K} A'_{(K-1)1}^\dagger \\
\bar{\mu}^{-1} A'_{11} & \frac{|\mu|^2}{1-|\mu|^2} b'_{22} & \bar{\mu}^{-1} A'_{31} & \frac{|\mu|^2}{1-|\mu|^2} A'_{41} & \cdots & c_{2K} A'_{K1} \\
\frac{1}{|\mu|^2-1} A'_{12} & \mu^{-1} A'_{22} & \frac{1}{|\mu|^2-1} b'_{33} & \mu^{-1} A'_{42} & \cdots & c_{3K} A'_{K2} \\
\bar{\mu}^{-1} A'_{13} & \frac{|\mu|^2}{1-|\mu|^2} A'_{23} & \bar{\mu}^{-1} A'_{33} & \frac{|\mu|^2}{1-|\mu|^2} b'_{44} & \cdots & c_{4K} A'_{K3} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
c_{K1} A'_{1(K-1)} & c_{K2} A'_{2(K-1)} & c_{K3} A'_{3(K-1)} & c_{K4} A'_{4(K-1)} & \cdots & c_{KK} b'_{KK}
\end{array} \right),
\end{aligned}$$

where

$$\begin{aligned}
a'_{ik} &= \left\langle \psi^{(i)} J^{-1} \mid J^{(k)} \right\rangle, \\
b'_{ik} &= \left\langle \psi^{(i)} J^{-1} \mid \psi^{(k)} \right\rangle, \\
A'_{ik} &= \left\langle \psi^{(i)} J^{-1} \mid A'_k \right\rangle, \\
A'^\dagger_{ik} &= \left\langle A'^\dagger_i J^{-1} \mid \psi^{(k)} \right\rangle, \\
\mu^{(1)} &= \mu, \\
\mu^{(i)} &= \frac{1}{\bar{\mu}^{(i-1)}} + \varepsilon \quad i = 2, 3, \dots, (K-1),
\end{aligned}$$

$$\begin{aligned}\left\langle \psi^{(i)} J^{-1} \mid \psi^{(j)} \right\rangle &= O(\varepsilon), \quad i \neq j, \\ A'_{i-1} &= \frac{\partial |\psi^{(i)}\rangle}{\partial \mu^{(i)}} \Big|_{\mu^{(i)}=\bar{\mu}^{-1}}.\end{aligned}$$

The values of coefficients  $c_{ij}$  are as given in equation (3.32).

## 6 Conclusions

We have constructed a binary Darboux transformation to generate exact multi-soliton solutions of  $U(N)$  principal chiral model. The multi-soliton solutions of the noncommutative  $U(N)$  principal chiral model are also obtained and the solutions are expressed in terms of quasi-determinants of Gel'fand and Retakh. We find that these solutions have the same form as that of (anti) self dual Yang-Mills equations. Our results are useful in the sense that their exact analysis leads to the various applications of D-brane dynamics and helps understanding the properties of  $N = 2$  string theory. This technique of binary Darboux transformation can also be applied to other integrable models to obtain their exact multi-soliton solutions. These solutions may be analysed and it would be interesting to check their stability and the scattering properties. The work can be further extended to construct super multi-solitons for the supersymmetric principal chiral model. From the point of view of string theory, it is also interesting to study the spectrum of solutions in the string theory on  $AdS^5 \times S^5$  using the binary Darboux transformation.

## Acknowledgements

BH would like to acknowledge the enabling role of the Higher Education Commission, Pakistan and appreciates its financial support through “Indigenous 5000 fellowship program” for PhD studies in Science and Technology. MH would like to thank Jonathan Nimmo for hospitality at the Department of Mathematics, University of Glasgow and to the Higher Education Commission, Pakistan for a research fellowship.

## References

- [1] K. Pohlmeyer, *Integrable hamiltonian systems and interactions through quadratic constraints*, Commun. Math. Phys. **46** (1976) 207.
- [2] V. E. Zakharov and A. V. Mikhailov, *Relativistically invariant two-dimensional models in field theory integrable by the inverse problem technique*, (In Russian), Sov. Phys. JETP **47** (1978) 1017 [Zh. Eksp. Teor. Fiz. **74**] (1978) 1953].
- [3] E. Brezin, C. Itzykson, J. Zinn-Justin and J. B. Zuber, *Remarks about the existence of nonlocal charges in two-dimensional models*, Phys. Lett. B **82** (1979) 442.
- [4] H. J. de Vega, *Field theories with an infinite number of conservation laws and Backlund transformations in two-dimensions*, Phys. Lett. B **87** (1979) 233.
- [5] S. Novikov, S. V. Manakov, L. P. Pitaevsky and V. E. Zakharov, *Theory of solitons. The inverse scattering method*, New York, Usa, Consultants Bureau ( 1984) 276 P. ( Contemporary Soviet Mathematics)
- [6] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian methods in the theory of solitons*, Berlin, Germany: Springer (1987) 592 p. (Springer series in Soviet Mathematics)
- [7] L. L. Chau, M. L. Ge, J. C. Shaw and H. C. Yen, *A unified derivation of Backlund transformations for integrable nonlinear equations*, Chinese Journal of Physics **5**, 34 (1996) 1191.
- [8] J. M. Evans, M. Hassan, N. J. MacKay and A. J. Mountain, *Local conserved charges in principal chiral models*, Nucl. Phys. B **561** (1999) 385
- [9] J. M. Evans, M. Hassan, N. J. MacKay and A. J. Mountain, *Conserved charges and supersymmetry in principal chiral and WZW models*, Nucl. Phys. B **580** (2000) 605.
- [10] U. Saleem and M. Hassan, *Zero-curvature formalism of supersymmetric principal chiral model*, Eur. Phys. J. C. **38**, (2005).
- [11] U. Saleem and M. Hassan, *Lax pair and Darboux transformation of noncommutative  $U(N)$  principal chiral model*, J. Phys. A: Math. Gen. **39** (2006).



- [12] J. Kluson, *Current algebra and integrability of principal chiral model on the world-sheet with general metric*, JHEP **704** (2007) 40 [arXiv:hep-th/0703003].
- [13] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the  $AdS^5 \times S^5$  superstring*, Phys. Rev. D **69** (2004) 046002 [arXiv:hep-th/0305116].
- [14] M. Hamanaka , *Notes on exact multi-soliton solutions of noncommutative integrable hierarchies* , JHEP **02** (2007) 094.
- [15] C. R. Gilson, J. J. C. Nimmo and Y. Ohta , *Quasideterminant solutions of a non-Abelian Hirota-Miwa equation*, J.Phys.A; Math.Theor. **40** (2007) 12607-12617.
- [16] C. R. Gilson and J. J. C. Nimmo, *On a direct approach to quasideterminant solutions of a noncommutative KP equation* , J.Phys.A; Math.Theor. **40** (2007) 3839-3850.
- [17] C. R. Gilson, J. J. C. Nimmo and C. M. Sooman , *On a direct approach to quasideterminant solutions of a noncommutative modified KP equation* , [nlin.SI / arXiv:0711.3733v1].
- [18] C. X. Li and J. J. C. Nimmo, *Quasideterminant solutions of a non-Abelian Toda lattice and kink solutions of a matrix sine-Gordon equation* , [nlin.SI / arXiv:0711.2594v2].
- [19] C. R. Gilson, M. Hamanaka and J. J. C. Nimmo , *Backlund transformations for noncommutative anti-self-dual Yang-Mills equations*, [nlin.SI / arXiv:0709.2069v1].
- [20] M. Siddiq, U. Saleem and M. Hassan, *Darboux transformation and multi-soliton solutions of a noncommutative sine-Gordon system* , To appear in Mod. Phys. Lett. A.
- [21] M. Hamanaka, *Commuting flows and conservation laws for noncommutative Lax hierarchies*, J. Math. Phys. **46**, (2005) 052701. (Preprint hep-th/0311206)
- [22] M. Hamanaka and K. Toda, *noncommutative Burgers equation*, J. Phys. A: Math. Gen. **36** (2003) 11981 (Preprint hep-th/0301213).
- [23] U. Saleem, M. Hassan and M. Siddiq, *Non-local continuity equations and binary Darboux transformation of noncommutative (anti) self-dual Yang-Mills equations*, J. Phys. A: Math. Theor. **40** (2007) 5205-5217.

- [24] M. Moriconi and I.C. Carnero, *noncommutative integrable field theories in 2-d*, Nucl. Phys. **B673** (2003) 437.
- [25] M.T. Grisaru and S. Penati, *An Integrable noncommutative version of the sine-Gordon system*, Nucl.Phys. **B655** (2003) 250.
- [26] K. Takasaki, *Anti-self-dual Yang-Mills equations on noncommutative spacetime*, J. Geom. Phys. **37**, 291 (2001).
- [27] U. Saleem, M. Hassan and M. Siddiq, *On noncommutative sinh-Gordon equation*, Chin. Phys. Lett. **22**, 1076 (2005).
- [28] U. Saleem, M. Hassan and M. Siddiq, *Conserved quantities in noncommutative principal chiral model with Wess-Zumino term*, J. Phys. A: Math.Gen. **38**, 9241 (2005).
- [29] A. Dimakis and F. Mueller Hoissen, *Extension of noncommutative soliton hierarchies*, J. Phys. A: Math. Gen. **37**, 4069 (2004).
- [30] A. Dimakis and F. M. Hoissen, *Bicomplexes, integrable models, and noncommutative geometry*, Int. J. Mod. Phys. **B14** (2000) 2455.
- [31] M. Hamanaka, *On reductions of noncommutative anti-self-dual Yang-Mills equations*, Phys. Lett. B **625**, 324 (2005). (Preprint hep-th/0507112)
- [32] M. Hamanaka, *noncommutative Ward's conjecture and integrable systems*, Nucl. Phys. B **741** 368 (2006) (Preprint hep-th/0601209)
- [33] S. Minwalla, M. V. Raamsdonk and N. Seiberg, *noncommutative perturbative dynamics*, J. High Energy Phys. **02** (2000) 020.
- [34] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, J. High Energy Phys. **09** (1999) 032.
- [35] K. Furuta and T. Inami, *Ultraviolet property of noncommutative Wess-Zumino-Witten model*, Mod. Phys. Lett. **A15** (2000) 997.

- [36] Gelfand and V. Retakh, *Determinants of matrices over noncommutative rings*, Funct. Anal. Appl. **25** (1991), no. 2, 91-102.
- [37] Gelfand and V. Retakh, *A theory of noncommutative determinants and characteristic functions of graphs*, Funct. Anal. Appl. **26** (1992), no. 4, 1-20.
- [38] I. Gelfand and V. Retakh, *A theory of noncommutative determinants and characteristic functions of graphs. I*, Publ. LACIM, UQAM, Montreal 14 (1993), 1-26.
- [39] Gelfand and V. Retakh, *Gelfand mathematical seminars 1993-95*, Birkhauser, 1996.
- [40] N. V. Ustinov, *The reduced selfdual Yang-Mills equation, binary and infinitesimal Darboux transformations*, J. Math. Phys. **39** (1998) 976.
- [41] J. E. Moyal, *Quantum mechanics as a statistical theory*, Proc. Camb. Phil. Soc. **45** (1949) 99.
- [42] N. Dorey and B. Vicedo, *A symplectic structure for string theory on integrable backgrounds*, JHEP **03** (2007) 045 [arXiv:hep-th/0606287].
- [43] G. Arutyunov and M. Zamaklar, *Linking Backlund and monodromy charges for strings on  $AdS^5 \times S^5$* , JHEP **0507** (2005) 026 [arXiv:hep-th/0504144].
- [44] L. F. Alday, G. Arutyunov and A. A. Tseytlin, *On integrability of classical superstrings in  $AdS^5 \times S^5$* , JHEP **0507** (2005) 002 [arXiv:hep-th/0502240].
- [45] A. Das, J. Maharana, A. Melikyan and M. Sato, *The algebra of transition matrices for the  $AdS^5 \times S^5$  superstring*, JHEP **0412** (2004) 055 [arXiv:hep-th/0411200].
- [46] G. Arutyunov and S. Frolov, *Integrable Hamiltonian for classical strings on  $AdS^5 \times S^5$* , JHEP **0502** (2005) 059 [arXiv:hep-th/0411089].
- [47] M. Hatsuda and K. Yoshida, *Classical integrability and super Yangian of superstring on  $AdS^5 \times S^5$* , Adv. Theor. Math. Phys. **9** (2005) 703 [arXiv:hep-th/0407044].
- [48] R. S. Ward, *On selfdual gauge fields*, Phys. Lett. A **61** (1977) 81.

- [49] R. S. Ward, *Integrable and solvable systems, and relations among them*, Phil. Trans. R. Soc. A **315** (1985) 451
- [50] R. S. Ward, *Multidimensional integrable systems*, (Lect. Notes Phys. vol 280) Springer p 106 (1986)